Augmented Lagrangian penalty method

input : starting point $\vec{x}_0$, initial multipliers $\vec{\lambda}_0 = \vec{0}$, initial penalty $\mu_0 > 0$, penalty increase factor $\rho > 1$, tolerance E_s

output: $\vec{x}^*$, estimate of the constrained minimizer, and $f(\vec{x}^*)$

$$\bar{g}_j(\vec{x}) = \begin{cases} h_j(\vec{x}) & \text{equality constraint } h_j = 0 \\ g_j(\vec{x}) & \text{inequality } g_j \le 0 \text{ with } g_j(\vec{x}) \ge -\lambda_j/\mu \\ -\lambda_j/\mu & \text{otherwise} \end{cases}$$

`// augmented Lagrangian (` $\vec{\lambda}$ `contains multipliers for both` $\vec{h}$ `and` $\vec{g}$`)`

$$\hat{f}(\vec{x};\, \vec{\lambda}, \mu) = f(\vec{x}) + \vec{\lambda}^{\mathsf{T}} \bar{g}(\vec{x}) + \frac{\mu}{2} \left\| \bar{g}(\vec{x}) \right\|_2^2$$

$k \leftarrow 0$

while $\|\bar{g}(\vec{x}_k)\| > E_s$ **do**

- $\vec{x}_k^* \leftarrow \underset{\vec{x}}{\text{minimize}}\ \hat{f}(\vec{x};\, \vec{\lambda}_k, \mu_k)$ `// unconstrained solve started from` $\vec{x}_k$
- $\vec{\lambda}_{k+1} \leftarrow \vec{\lambda}_k + \mu_k\, \bar{g}(\vec{x}_k^*)$ `// update multipliers`
- $\mu_{k+1} \leftarrow \rho\, \mu_k$ `// increase penalty`
- $\vec{x}_{k+1} \leftarrow \vec{x}_k^*$ `// next starting point`
- $k \leftarrow k + 1$

end

return $\vec{x}^* = \vec{x}_k,\ \ f(\vec{x}^*)$