Today

Boundary Value Problems

Shooting Method

Finite Difference

Announcements

HW5 due Thursday (no late penalty)

## So far: initial value problems

$$\vec{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \qquad \dot{\vec{x}} = f(t, \vec{x}) \qquad \begin{matrix} x(0) = a \\ \dot{x}(0) = b \end{matrix}$$ ← initial conditions

## Boundary-value problems

Example: $x(0) = a$ (initial pos.) $\qquad x(T) = b$ (final pos.)

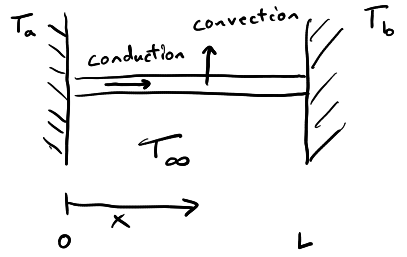
$\dot{x}(0), \dot{x}(T)$ unknown

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$x$ known: Dirichlet B.C. $\qquad$ $\dot{x}$ known: Neumann B.C.

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Example: heated rod




$$\frac{d^2T}{dx^2} + h'(T_\infty - T) + \sigma'(T_\infty - T) = 0 \qquad T(0) = T_a, \; T(L) = T_b$$

($\sigma'(T_\infty - T)$: radiation)

Solve analytically

$$\vec{y} = \begin{bmatrix} T \\ \frac{dT}{dx} \end{bmatrix} \qquad \frac{d\vec{y}}{dx} = f(t, \vec{y})$$

$$\vec{y}(0) = \begin{bmatrix} T(0) = T_a \\ \frac{dT}{dx}(0) \end{bmatrix} \qquad \text{ode45}(f, [0, L], \vec{y}(0))$$

Start with $\frac{T_b - T_a}{L}$, refine

## Shooting Method

Start with known IC's, guess unknown ICs

Adjust unknown IC's and "shoot" until we satisfy all BCs

How do we shoot in a smart way?

Let $z \equiv \frac{dT}{dx}$ $\qquad z(0)$ is unknown IC

define $g(T(0), z(0))$ integrates (using RK4/ode45) and returns $T(L)$

$g(T_a, z(0)) = T_b \Rightarrow 0 = g(T_a, z(0)) - T_b$ $\qquad f(z(0)) = 0$

($z(0)$: unknown) $\qquad$ nonlinear scalar equation

solve for $z(0)$ using _____

↑ your favorite nonlinear solver

Newton Raphson, Incremental Search, Secant Bisection

can diverge for complex dynamics

# Finite Difference Method

1) Discretize problem into $n$ $\Delta x$ segments
2) Replace derivatives with finite difference (central finite difference)
3) solve resulting equations

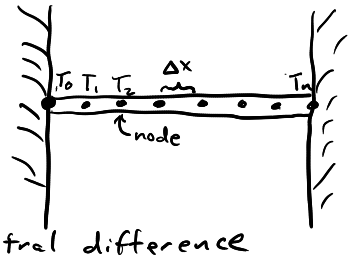
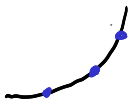



Fig 21.5: central difference

| First Derivative | Error |
|---|---|
| $f'(x_i)=\frac{f(x_{i+1})-f(x_{i-1})}{2h}$ | $O(h^2)$ |
| $f'(x_i)=\frac{-f(x_{i+2})+8f(x_{i+1})-8f(x_{i-1})+f(x_{i-2})}{12h}$ | $O(h^4)$ |
| **Second Derivative** | |
| $f''(x_i)=\frac{f(x_{i+1})-2f(x_i)+f(x_{i-1})}{h^2}$ | $O(h^2)$ |
| $f''(x_i)=\frac{-f(x_{i+2})+16f(x_{i+1})-30f(x_i)+16f(x_{i-1})-f(x_{i-2})}{12h^2}$ | $O(h^4)$ |
| **Third Derivative** | |
| $f'''(x_i)=\frac{f(x_{i+2})-2f(x_{i+1})+2f(x_{i-1})-f(x_{i-2})}{2h^3}$ | $O(h^2)$ |
| $f'''(x_i)=\frac{-f(x_{i+3})+8f(x_{i+2})-13f(x_{i+1})+13f(x_{i-1})-8f(x_{i-2})+f(x_{i-3})}{8h^3}$ | $O(h^4)$ |
| **Fourth Derivative** | |
| $f''''(x_i)=\frac{f(x_{i+2})-4f(x_{i+1})+6f(x_i)-4f(x_{i-1})+f(x_{i-2})}{h^4}$ | $O(h^2)$ |
| $f''''(x_i)=\frac{-f(x_{i+3})+12f(x_{i+2})-39f(x_{i+1})+56f(x_i)-39f(x_{i-1})+12f(x_{i-2})-f(x_{i-3})}{6h^4}$ | $O(h^4)$ |

$$\frac{d^2T}{dx^2} \approx \frac{T_{i-1}-2T_i+T_{i+1}}{\Delta x^2}$$



$$\frac{d^2T}{dx^2}+h'(T_\infty-T)=0$$

$$\frac{\bar{T}_{i-1}-2\bar{T}_i+\bar{T}_{i+1}}{\Delta x^2}+h'(T_\infty-\bar{T}_i)=0$$

$$-T_{i-1}+(2+h'\Delta x^2)T_i-T_{i+1}=h'\Delta x^2 T_\infty$$

$T_0=T_a \qquad T_n=T_b$

case: $L=10m \qquad h'=0.05\,m^{-2}$

$T_\infty=200K \qquad T_a=300K,\ T_b=400K$

$i=1$: $-T_a+(2+h'\Delta x^2)T_1-T_2=h'\Delta x^2 T_\infty+T_a$

$i=2$: $-T_1+(2+h'\Delta x^2)T_2-T_3=h'\Delta x^2 T_\infty$

$\vdots$

$i=n-1$: $-T_{n-2}+(2+h'\Delta x^2)T_{n-1}-T_n=h'\Delta x^2 T_\infty+T_b$

plug in numbers and put into matrix form:

$2+h'\Delta x^2=2.2,\ h'\Delta x^2 T_\infty=40,\ n=5$

$$\begin{bmatrix} 2.2 & -1 & 0 & 0 \\ -1 & 2.2 & -1 & 0 \\ 0 & -1 & 2.2 & -1 \\ 0 & 0 & -1 & 2.2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 40+300 \\ 40 \\ 40 \\ 40+400 \end{bmatrix}$$

→ Solve with linear solver.

Since this is tridiagonal, you can use an optimized $O(n)$ solver.

→ If the equations were nonlinear, you could use Newton-Raphson for systems. The Jacobian would be tridiagonal.