## Today

Systems of nonlinear equations

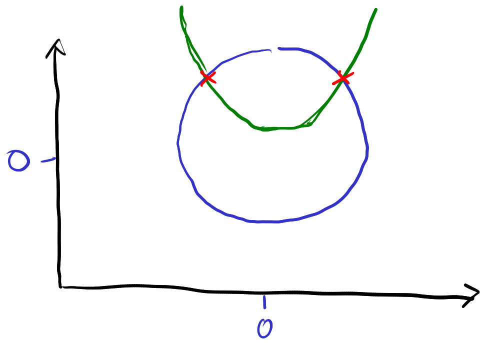
Newton-Raphson for systems

Numerical differentiation

## Announcements

- HW 3 due Thurs
- Exam Next Monday
  - Will release Eqn. Sheet/ practice exam today/tomorrow
  - Calculator
  - Study HW, Labs, Notes

Up to now $f(x) = 0$




$$x_1^2 + x_2^2 = 4$$

$$x_2 = x_1^2$$

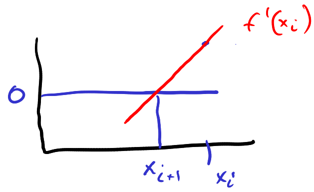
$$\{f\}(\{x\}) = \{0\}$$

$$\vec{f}(\vec{x}) = \vec{0}$$

$$\vec{f}(\vec{x}) = \begin{bmatrix} x_1^2 + x_2^2 - 4 \\ x_2 - x_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Newton - Raphson for scalar:

$$f(x_i) = f(x_{i+1}) + f'(x_i)(x_i - x_{i+1})$$




vector case:

first element of f →

choose $\vec{x}_{i+1}$ so this is zero

$$f_1(\vec{x}_i) = \cancel{f_1(\vec{x}_{i+1})} + \frac{\partial f_1}{\partial x_1}(x_{1,i} - x_{1,i+1}) + \frac{\partial f_1}{\partial x_2}(x_{2,i} - x_{2,i+1}) + \dots \frac{\partial f_1}{\partial x_n}(x_{n,i} - x_{n,i+1})$$

$$= \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j}(x_{j,i} - x_{j,i+1})$$

$$f_2(\vec{x}_i) = \sum_{j=1}^{n} \frac{\partial f_2}{\partial x_j}(x_{j,i} - x_{j,i+1})$$

$$\vdots$$

$$f_n(\vec{x}_i) = \sum_{j=1}^{n} \frac{\partial f_n}{\partial x_j}(x_{j,i} - x_{j,i+1})$$

$$\vec{f}(\vec{x}_i) = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}}_{J} \left( \begin{bmatrix} x_{1,i} \\ \vdots \\ x_{n,i} \end{bmatrix} - \begin{bmatrix} x_{1,i+1} \\ \vdots \\ x_{n,i+1} \end{bmatrix} \right)$$

$$\underbrace{\vec{f}(\vec{x}_i)}_{\text{known}} = \underbrace{J}_{\text{known}} (\underbrace{\vec{x}_i}_{\text{known}} - \underbrace{\vec{x}_{i+1}}_{\text{unknown}})$$

$$J^{-1}\vec{f}(\vec{x}_i) = \vec{x}_i - \vec{x}_{i+1}$$

$$\boxed{\vec{x}_{i+1} = \vec{x}_i - J^{-1}\vec{f}(\vec{x}_i)}$$

(scalar)

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

want $\vec{x}_{i+1} = \vec{x}_i + \Delta\vec{x}$

$\Delta\vec{x} = \vec{x}_{i+1} - \vec{x}_i$

$$\boxed{\begin{array}{l}\text{solve } J\,\Delta\vec{x} = -\vec{f}(\vec{x}_i) \\ \vec{x}_{i+1} \leftarrow \vec{x}_i + \Delta\vec{x}\end{array}}$$

or

$$\boxed{\vec{x}_{i+1} \leftarrow \vec{x}_i - J\backslash\vec{f}(\vec{x}_i)}$$

$A\backslash b \equiv$ solution to $Ax=b$

**Newton-Raphson for Systems**

**input** : function $\{f\}: \mathbb{R}^n \to \mathbb{R}^n$ and its Jacobian $[J]$ (with $J_{ij} = \partial f_i/\partial x_j$), initial guess $\{x\}_0$, tolerance $\varepsilon_s$, maximum iterations $N$

**output:** estimate $\{x\}_r$ of the root

$\{x\}_r \leftarrow \{x\}_0$

**for** $i \leftarrow 1$ **to** $N$ **do**

  $\{x\}_{r,\text{old}} \leftarrow \{x\}_r$

  solve $[J](\{x\}_{r,\text{old}})\,\{\Delta x\} = -\{f(\{x\}_{r,\text{old}})\}$

  $\{x\}_r \leftarrow \{x\}_{r,\text{old}} + \{\Delta x\}$

  $\varepsilon_a \leftarrow \dfrac{\|\{x\}_r - \{x\}_{r,\text{old}}\|}{\|\{x\}_r\|} \cdot 100\%$

  **if** $\varepsilon_a < \varepsilon_s$ **then**

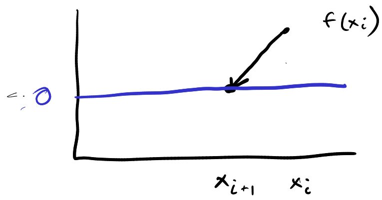
    **return** $\{x\}_r$

  **end**

**end**

**return** $\{x\}_r$ `// did not converge in N iterations`

(scalar)




## Two Challenges

1. Convergence
2. Calculating Jacobian

## Strength

"quadratic convergence" near root

## Optimization

minimize $\sum_{k=1}^{n} \left(f_k(\vec{x})\right)^2$

## Calculating Derivatives

**Symbolic**
- By hand
- Computer Algebra System (CAS)
  - e.g. Symbolic Toolbox
  - Automatic Code gen

**Numerical**

**Automatic**
- Tracks exact derivative through each operation

# Numerical Differentiation

Forward: $f'(x) \approx \dfrac{f(x+h) - f(x)}{h}$

Central: $f'(x) \approx \dfrac{f(x+h) - f(x-h)}{2h}$ ← more accurate

Column of Jacobian → $\dfrac{\partial \vec{f}}{\partial x_j} \approx \dfrac{\vec{f}(\vec{x} + h\vec{e}_j) - \vec{f}(\vec{x})}{h}$

size of $h$?

$h \approx \sqrt{\varepsilon}\, x$

↑ machine precision eps

**Numerical Jacobian (forward difference)**

**input** : function $\{f\} : \mathbb{R}^n \to \mathbb{R}^n$, point $\{x\}$, step size $h$
**output:** approximation $[J]$ of the Jacobian at $\{x\}$, $J_{ij} \approx \partial f_i / \partial x_j$

$\{f\}_0 \leftarrow \{f(\{x\})\}$
**for** $j \leftarrow 1$ **to** $n$ **do**
  $\{e_j\} \leftarrow$ unit vector in the $j$ direction  `//` $j$`th column of` $[I]$
  $[J][:, j] \leftarrow \dfrac{\{f(\{x\} + h\,\{e_j\})\} - \{f\}_0}{h}$  `//` $j$`th column of` $[J]$
**end**
**return** $[J]$