Today

Open Methods: Newton-Raphson + Friends

Secant, Brent's, Fixed Point Iter (maybe)

Reminders

Lab 1 due tomorrow

First: Errors (Chapra's notation)

E : absolute error (units)

ε : relative (percent) error

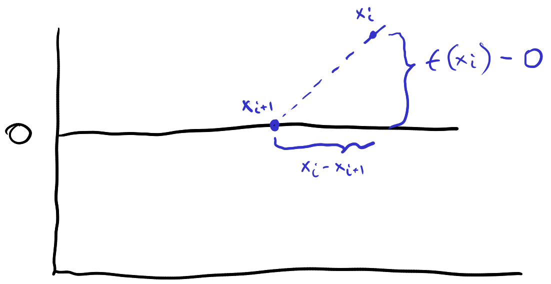
Subscripts

t : true (typically unknown)

a : approximate

s : stopping

Newton-Raphson : use $f'(x)$



$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}} \qquad \frac{\text{rise}}{\text{run}}$$

$$(x_i - x_{i+1})\, f'(x_i) = f(x_i)$$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

Newton-Raphson Most widely used

input : function f and its derivative f', initial guess x_0, tolerance ε_s, maximum iterations N

output: estimate x_r of the root

$x_r \leftarrow x_0$

for $i \leftarrow 1$ **to** N **do**

  $x_{r,\text{old}} \leftarrow x_r$

  $x_r \leftarrow x_{r,\text{old}} - \dfrac{f(x_{r,\text{old}})}{f'(x_{r,\text{old}})}$ ← update

  $\varepsilon_a \leftarrow \left|\dfrac{x_r - x_{r,\text{old}}}{x_r}\right| \cdot 100\%$

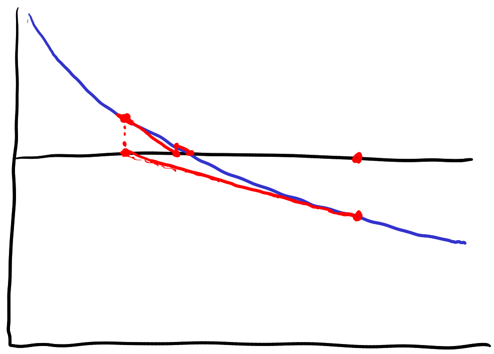
  if $\varepsilon_a < \varepsilon_s$ **then**

    return x_r

  end

end

return x_r `// did not converge in N iterations`



Beauty of Newton-Raphson: Extremely fast convergence near root!

Bisection

$$E_{a,i+1} = \tfrac{1}{2} E_{a,i}$$

"linear convergence"

number of sig figs increases linearly with i

(different terminology from $E_{a,i} = O\left(\left(\tfrac{1}{2}\right)^k\right)$)

Gauranteed to converge if

- Δx contains root
- $f(x)$ is continuous

fast

Newton Raphson

Near Root "can be shown"

$$\rightarrow E_{t,i+1} \simeq -\frac{f''(x_r)}{2f'(x_r)} E_{t,i}^2$$

"quadratic convergence"

number of sigfigs doubles every iteration

Can diverge

FAST near root

Downside: Newton requires $f'(x)$

Secant Method

$$f'(x_i) \approx \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

plug into Newton update

$$x_{i+1} = x_i - \frac{f(x_i)\,(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

Secant Method

input : function f, initial guesses x_{-1} and x_0, tolerance ε_s, maximum iterations N

output: estimate x_r of the root

$x_{\text{prev}} \leftarrow x_{-1}$; $\quad x_r \leftarrow x_0$

for $i \leftarrow 1$ **to** N **do**

$\quad x_{r,\text{old}} \leftarrow x_r$

$\quad x_r \leftarrow x_{r,\text{old}} - \dfrac{f(x_{r,\text{old}})\,(x_{\text{prev}} - x_{r,\text{old}})}{f(x_{\text{prev}}) - f(x_{r,\text{old}})}$

only difference from Newton

$\quad x_{\text{prev}} \leftarrow x_{r,\text{old}}$

$\quad \varepsilon_a \leftarrow \left| \dfrac{x_r - x_{r,\text{old}}}{x_r} \right| \cdot 100\%$

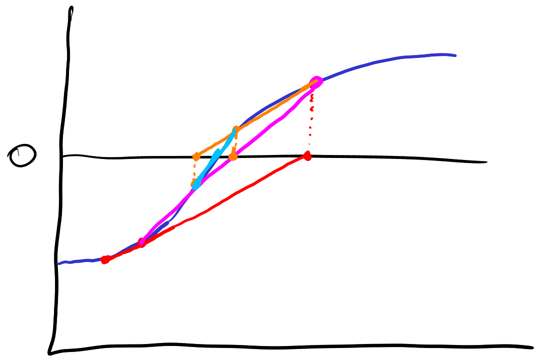
$\quad$ **if** $\varepsilon_a < \varepsilon_s$ **then**

$\quad\quad$ **return** x_r

$\quad$ **end**

end

return x_r `// did not converge in N iterations`



Brent's Method

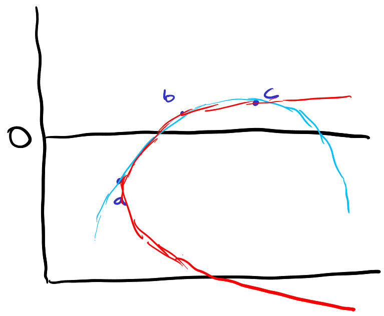
- Combination of Bracketing and Open Methods
- Does not require derivatives
- Basis for Matlab's fzero

- Gauranteed to converge
- As fast near root as open methods

Brent's Method

input : function f, bracket $[a, b]$ with $f(a) \cdot f(b) < 0$, tolerance ε_s
output: estimate b of the root

// b is the best estimate, c is on the other side of the root, a is the previous b
$c \leftarrow a$
// d is the signed step b will move by, e is the previous step; both start as the bracket width
$d \leftarrow b - a$; $e \leftarrow d$
repeat
  if $f(b) \cdot f(c) > 0$ **then**
    // b and c no longer bracket the root; a does
    $c \leftarrow a$; $d \leftarrow b - a$; $e \leftarrow d$
  end
  if $|f(c)| < |f(b)|$ **then**
    // make b the point with the smallest $|f|$
    $a \leftarrow b$; $b \leftarrow c$; $c \leftarrow a$
  end
  $m \leftarrow (c - b) / 2$ // bisection step ← bisection
  $\delta \leftarrow \varepsilon_s |b|$ // absolute tolerance
  if $|m| \leq \delta$ **or** $f(b) = 0$ **then**
    return b
  end
  if $|e| < \delta$ **or** $|f(a)| \leq |f(b)|$ **then**
    $d \leftarrow m$; $e \leftarrow m$ // bisect
  else
    $s \leftarrow f(b) / f(a)$
    if $a = c$ **then** ← secant
      // only two distinct points: secant step
      $p \leftarrow 2ms$; $q \leftarrow 1 - s$
    else ← inverse quadratic
      // three distinct points: inverse quadratic interpolation
      $q \leftarrow f(a) / f(c)$; $r \leftarrow f(b) / f(c)$
      $p \leftarrow s\,(2mq\,(q - r) - (b - a)(r - 1))$
      $q \leftarrow (q - 1)(r - 1)(s - 1)$
    end
    if $p > 0$ **then** $q \leftarrow -q$
    else $p \leftarrow -p$
    if $2p < 3mq - |\delta q|$ **and** $p < |eq/2|$ **then**
      // interpolated step stays in the bracket and shrinks it fast enough
      $e \leftarrow d$; $d \leftarrow p / q$
    else
      $d \leftarrow m$; $e \leftarrow m$ // bisect
    end
  end
  $a \leftarrow b$



Fixed Point

Previously $f(x) = 0$

Fixed Point $\boxed{x = g(x)}$

repeatedy evaluate $g(x)$