Matrix Inverse
Condition Number
Roundoff Error

Homework 2 Due Thursday

$$Ax = b$$

inverse $\rightarrow$ $A^{-1}b = x$

$$A A^{-1} b = Ax = b$$

$$A A^{-1} b = b$$

$$\underset{(3\times 3)}{A A^{-1} = I}$$

$$A \begin{bmatrix} a^{-1}_{11} \\ a^{-1}_{21} \\ a^{-1}_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} a^{-1}_{12} \\ a^{-1}_{22} \\ a^{-1}_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} a^{-1}_{13} \\ a^{-1}_{23} \\ a^{-1}_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Matrix Inverse via LU factorization

input : $n \times n$ matrix $[A]$
output: $[A]^{-1}$

$[L], [U] \leftarrow$ `lu_decompose`$([A])$
for $k \leftarrow 1$ **to** n **do**
  $\{e_k\} \leftarrow$ unit vector with 1 at element k; 0 elsewhere
  $A^{-1}[:, k] \leftarrow$ `lu_solve`$([L], [U], \{e_k\})$
end

What can go wrong? from APPM 2360

$Ax=b$ does not have a unique solution $\Longleftrightarrow$ A is singular $\Longleftrightarrow$ $\det(A) = 0$ $\Longleftrightarrow$ A is not full rank $\Longleftrightarrow$ A^{-1} does not exist

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$
not full rank

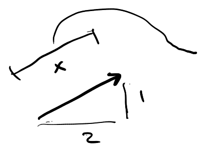
What if A is almost singular?

$$\boxed{\text{cond}(A) = \|A\| \cdot \|A^{-1}\|}$$

large cond $\Rightarrow$ nearly singular

$\| \ \|$ means "norm" Book covers many types of norms

For now $\| \ \| = \| \ \|_2$ "2-norm"



$\|x\|_2 = \sqrt{1^2+2^2}$

For a vector x, $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \dots x_n^2}$, "length" of x

For a matrix A, $\|A\|_2 = \max\limits_{x \neq 0} \dfrac{\|Ax\|_2}{\|x\|_2} = \sqrt{\mu_{max}}$

"The most A can elongate a vector" max eigenvalue

Key property of cond:

$$\boxed{\frac{\|\Delta x\|}{\|x\|} \leq \text{cond}(A) \frac{\|\Delta A\|}{\|A\|}}$$

relative error in result relative error in matrix

Roundoff Errors

Computer uses only 3 significant figures

747 takeoff mass

3.97×10^5 kg   burns 4.21 kg/s fuel

$$\begin{array}{r l} 397{,}000 & \text{kg} \\ -\quad 4.21 & \text{kg} \\ \hline 396{,}995.71 & \end{array} \longrightarrow 3 \text{ sig figs} \longrightarrow 3.97 \times 10^5 \text{ kg}$$

How computers store numbers: bits (1 or 0)

Humans think in decimal digits
0, …, 9

Computers think in Binary
0, 1

Unsigned Integers

	1000	100	10	1
3	0	0	0	3
137	0	1	3	7
43	0	0	4	3

1 byte

128	64	32	16	8	4	2	1
0	0	0	0	0	0	1	1
1	0	0	0	1	0	0	1
0	0	1	0	1	0	1	1

Scientific Notation

$\pm s \times 10^{e}$

exponent: e; significand: s

Floating Point
IEEE 754

$\pm (1+f) \times 2^{(e-127)}$

exponent: $(e-127)$; mantissa: f

sign	exponent	mantissa

precision	sign	exponent	mantissa	
single	1	8	23 ≈ 7 decimal digits	= 32 bits
double	1	11	52 ≈ 15 decimal digits	= 64 bits

matlab/Julia eps(x) distance to next floating point number above x

$$\frac{\|\Delta A\|}{\|A\|} \approx \text{eps}(1.0)$$

$$\boxed{\frac{\|\Delta x\|}{\|x\|} \lesssim \text{cond}(A)\, \varepsilon}$$