Today

LU-Factorization

HW1 Due tomorrow
HW2 Assigned tonight
Lab 1: catch up with team to
1.3 (minimum)
1.5 (recommended)
before Friday

Study Hall tonight!

Recall: with a triangular matrix, T, solving $Tx=b$ for x is $O(n^2)$

Suppose we have

$$\underset{\text{lower triangular}}{L}\ \underset{\text{upper triangular}}{U} = A$$

$$Ax = b$$
$$L\underbrace{Ux}_{d} = b$$

0. factorize A into LU
1. solve $Ld = b$
2. solve $U\underline{x} = d$

both steps solve with a triangular matrix

$$\Rightarrow L \qquad\qquad \Rightarrow U$$

$$\left(\underset{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}{I} + \begin{bmatrix}0&0&0\\2&0&0\\0&0&0\end{bmatrix}\right)\begin{bmatrix}2&3&4\\0&5&6\\2&8&17\end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix}2&3&4\\4&11&14\\2&8&17\end{bmatrix}$$

$$\left(I + \begin{bmatrix}0&0&0\\2&0&0\\1&0&0\end{bmatrix}\right)\begin{bmatrix}2&3&4\\0&5&6\\0&5&13\end{bmatrix} \overset{\checkmark}{=} \quad ''$$

$$\left(I + \begin{bmatrix}0&0&0\\2&0&\\1&1&0\end{bmatrix}\right)\begin{bmatrix}2&3&4\\0&5&6\\0&0&7\end{bmatrix} \overset{\checkmark}{=} \quad ''$$

$$\underset{L}{\begin{bmatrix}1&0&0\\2&1&0\\1&1&1\end{bmatrix}}\underset{U}{\begin{bmatrix}2&3&4\\0&5&6\\0&0&7\end{bmatrix}} = \begin{bmatrix}2&3&4\\4&11&14\\2&8&17\end{bmatrix}$$

$$L = \begin{bmatrix}1 & & & & \\ f_{21} & 1 & & & \\ f_{31} & f_{32} & \ddots & & \\ \vdots & & \ddots & \ddots & \\ f_{n1} & & & f_{n,n-1} & 1\end{bmatrix}$$

U = result of GE

f_{ij} = factor from GE for eliminating element ij

LU factorization: run GE and save factors in L

$$A = \begin{bmatrix}2&1\\6&5\end{bmatrix}\begin{matrix}①\\②\end{matrix}$$ subtract $\underline{3} \times ①$ from 2

$$L = \begin{bmatrix}1&0\\3&1\end{bmatrix} \qquad U = \begin{bmatrix}2&1\\0&2\end{bmatrix} \qquad LU = \begin{bmatrix}2&1\\6&5\end{bmatrix} \overset{\checkmark}{=} A$$

$$\begin{tikzcd}[column sep=small]
 & & Ax = b \arrow[dll] \arrow[d] \arrow[dr, dashed] \arrow[ddr] & \\
U \arrow[ddd] & & L \arrow[d] & P \arrow[d, dashed] \\
 & & Ld = Pb \arrow[d] & \\
 & & & \\
Ux = d \arrow[d] & & & \\
\boxed{x} & & &
\end{tikzcd}$$

$Ax=b$ } Factorization $O(n^3)$ — only involves A, not b!

$Ld = Pb$ } (Forward) Substitution $O(n^2)$

$Ux = d$ } (Back) Substitution $O(n^2)$

If we have many b's to solve for, factorize once $\big(O(n^3)\big)$ then do forward and back substitution for each b $\big(O(n^2)\big)$!

What about Pivoting?

$LU = PA$

↑ permutation matrix (keep track of pivots)

$$\underbrace{\begin{bmatrix} 4 & 11 & 14 \\ 2 & 3 & 4 \\ 2 & 8 & 17 \end{bmatrix}}_{A'} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 2 & 3 & 4 \\ 4 & 11 & 14 \\ 2 & 8 & 17 \end{bmatrix}}_{A}$$

each row i determines which row j of A ends up in A'

$p_{ij} = 1$ if row j of A becomes row i of A'

Exercise 2

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$Ax = b$

$LU = PA$

$PAx = Pb$

$LUx = Pb$

$$\begin{bmatrix} - c - \\ - b - \\ - a - \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$