Counting the Operations in Gaussian Elimination: Detailed Derivation

ASEN 3502 — Computational Methods

In class we counted the floating point operations ("flops") required by Gaussian elimination and found that $\frac{2n^3}{3} + O(n^2)$. That derivation moved quickly. These notes provide a step-by step derivation of the trickiest line (5).

1 The algorithm

Here is the same Gaussian elimination algorithm from the slides. Every `for` loop is written out, and each line is numbered so we can refer to it below.

input : $n \times n$ matrix $[A]$, right-hand side $\{b\}$
output: solution $\{x\}$ of $[A]\{x\} = \{b\}$

```
// forward elimination
1  for c ← 1 to n − 1 do
2  |  for r ← c + 1 to n do
3  |  |  f ← A[r, c] / A[c, c]
4  |  |  for j ← c to n do
5  |  |  |  A[r, j] ← A[r, j] − f · A[c, j]
6  |  |  end
7  |  |  b[r] ← b[r] − f · b[c]
8  |  end
9  end
// back substitution
10 for r ← n, n − 1, . . . , 1 do
11 |  s ← b[r]
12 |  for j ← r + 1 to n do
13 |  |  s ← s − A[r, j] · x[j]
14 |  end
15 |  x[r] ← s / A[r, r]
16 end
```

2 Number of executions of line 5

This derivation will focus on finding the **number of times line 5 is executed**. We'll call this number N_5.

Since line 5 is within three nested for loops. We need to evaluate the triple sum

$$N_5 = \sum_{c=1}^{n-1} \sum_{r=c+1}^{n} \sum_{j=c}^{n} 1. \tag{1}$$

The summand is 1 because we are counting executions: each combination of (c, r, j) contributes one execution of line 5.

2.1 Sum identities

The following sum identities are used in the derivation:

$$\sum_{i=k}^{m} a\, f(i) = a \sum_{i=k}^{m} f(i) \tag{ID1}$$

$$\sum_{i=k}^{m} [f(i) + g(i)] = \sum_{i=k}^{m} f(i) + \sum_{i=k}^{m} g(i) \tag{ID2}$$

$$\sum_{i=1}^{m} 1 = m \tag{ID3}$$

$$\sum_{i=k}^{m} 1 = m - k + 1 \tag{ID4}$$

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2} \tag{ID5}$$

$$\sum_{i=1}^{m} i^2 = \frac{m(m+1)(2m+1)}{6} \tag{ID6}$$

The proofs of these are not a focus of the class, but I encourage you to look them up if you are curious! Asking an LLM is the easiest way to do this.

2.2 Step-by-step derivation

$$\begin{aligned}
\sum_{c=1}^{n-1}\sum_{r=c+1}^{n}\sum_{j=c}^{n}1 &= \sum_{c=1}^{n-1}\sum_{r=c+1}^{n}(n-c+1) && \text{(using ID4)}\\
&= \sum_{c=1}^{n-1}(n-c+1)\sum_{r=c+1}^{n}1 \\
& && \text{(using ID1. Note: } n \text{ and } c \text{ are constants within the } r \text{ sum.)}\\
&= \sum_{c=1}^{n-1}(n-c+1)(n-c) && \text{(using ID4 again)}\\
&= \sum_{c=1}^{n-1}\left[n^2-cn-cn+c^2+n-c\right] && \text{(multiplying terms)}\\
&= \sum_{c=1}^{n-1}\left[c^2-c(2n+1)+n^2+n\right] && \text{(collecting terms by the sum counter, } c\text{)}\\
&= \sum_{c=1}^{n-1}c^2+\sum_{c=1}^{n-1}\left[-c(2n+1)\right]+\sum_{c=1}^{n-1}\left[n^2+n\right] && \text{(using ID2)}\\
&= \sum_{c=1}^{n-1}c^2-(2n+1)\sum_{c=1}^{n-1}c+(n^2+n)\sum_{c=1}^{n-1}1 \\
& && \text{(using ID1. Note: } n \text{ is a constant within the sum.)}\\
&= \frac{(n-1)(n-1+1)(2(n-1)+1)}{6}-(2n+1)\frac{(n-1)(n-1+1)}{2}+(n^2+n)(n-1) \\
& && \text{(using ID6, ID5, and ID3)}\\
&= \frac{2n^3-3n^2+n}{6}-\frac{2n^3-n^2-n}{2}+n^3-n \\
N_5 &= \frac{n^3}{3}-\frac{n}{3}
\end{aligned}$$

3 Adding up all of the flops

The following table shows the number of flops for each line and number of times each line is executed:

Line	Operation	Flops	Times executed
1	**for** $c \leftarrow 1$ **to** $n-1$	0	$n-1$
2	**for** $r \leftarrow c+1$ **to** n	0	$\frac{n(n-1)}{2}$
3	$f \leftarrow A[r,c] \,/\, A[c,c]$	1	$\frac{n(n-1)}{2}$
4	**for** $j \leftarrow c$ **to** n	0	$\frac{n^3-n}{3}$
5	$A[r,j] \leftarrow A[r,j] - f \cdot A[c,j]$	2	$\frac{n^3-n}{3}$
7	$b[r] \leftarrow b[r] - f \cdot b[c]$	2	$\frac{n(n-1)}{2}$
10	**for** $r \leftarrow n, n-1, \ldots, 1$	0	n
11	$s \leftarrow b[r]$	0	n
12	**for** $j \leftarrow r+1$ **to** n	0	$\frac{n(n-1)}{2}$
13	$s \leftarrow s - A[r,j] \cdot x[j]$	2	$\frac{n(n-1)}{2}$
15	$x[r] \leftarrow s \,/\, A[r,r]$	1	n

Multiplying the number of flops by the number of times executed and adding them all up results in

$$\frac{2}{3}n^3 + \frac{5}{2}n^2 - \frac{13}{6}n = \frac{2}{3}n^3 + O(n^2)$$

total flops.