Today

8/26

- Linear Systems of Equations
- Gaussian Elim.
- Partial Pivoting

$$2x_1 + x_2 = 4$$
$$-x_1 + x_2 = 1$$

Unkowns: $x_1, \dots, x_n$

Linear Eqn.: $\sum_{j=1}^{n} a_j x_j = b$

no nonlinear functions of x_j: x^2, e^x, $\log(x)$, $\sin(x)$ etc.

System of m linear equations: $\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1 \dots m$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Book: $[A]\{x\} = \{b\}$

for now $m = n$

Handwriting: $A\vec{x} = \vec{b}$

HWO $\Rightarrow A \cdot x$

Lab $\quad Ax = b$

$\underset{\text{known}}{A}\;\underset{\text{unknowns}}{x} = \underset{\text{known}}{b}$

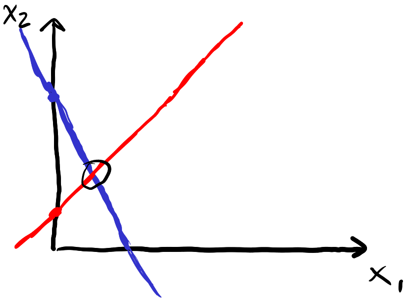
$2x_1 + x_2 = 4$

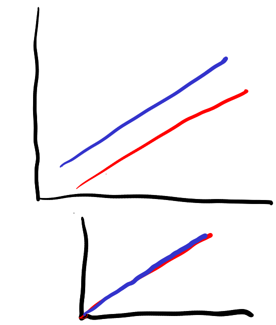
$x_2 = -2x_1 + 4$

$-x_1 + x_2 = 1$

$x_2 = x_1 + 1$

$\boxed{x_1 = 1, \; x_2 = 2}$





Example:

$$\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$3x_1 - x_2 = 0 \rightarrow 3x_1 - 3 = 0$

$\boxed{x_1 = 1}$

$2x_2 = 6$

$\boxed{x_2 = 3}$

$$
\begin{aligned}
2x_1 + 3x_2 + 4x_3 &= 19 \quad (1)\\
4x_1 + 11x_2 + 14x_3 &= 55 \quad (2)\\
2x_1 + 8x_2 + 17x_3 &= 50 \quad (3)
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c} 2 & 3 & 4 & 19 \\ 4 & 11 & 14 & 55 \\ 2 & 8 & 17 & 50 \end{array}\right]
\quad
\begin{array}{c} f \\ -2\cdot(1) \\ -1\cdot(1) \end{array}
\quad
\left[\begin{array}{ccc|c} 2 & 3 & 4 & 19 \\ 0 & 5 & 6 & 17 \\ 0 & \boxed{5} & 13 & 31 \end{array}\right]
$$

$$
-1\cdot(2) \quad
\left[\begin{array}{l} 2x_1 + 3x_2 + 4x_3 = 19 \\ 0 \quad 5x_2 + 6x_3 = 17 \\ 0 \quad 0 \quad \underbrace{7x_3 = 14} \end{array}\right]
\qquad \boxed{x_3 = 2}
$$

$2x_1 + 3\cdot 1 + 4\cdot 2 = 19$

$2x_1 = 8$

$\boxed{x_1 = 4}$

$5x_2 + 12 = 17$

$\boxed{x_2 = 1}$

Eliminate [blue] with [cyan]

Eliminate [red] with [magenta]

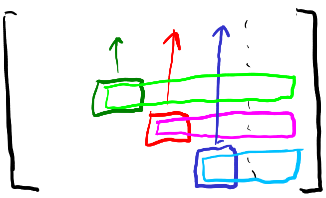
etc

Example:

$$
A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 9 & -2 \\ 2 & 1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 9 \\ 2 \end{bmatrix}
$$

$$
x = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}
$$

Back substitution



Solve for [blue] with [cyan]

Solve for [red] with [magenta]

etc.

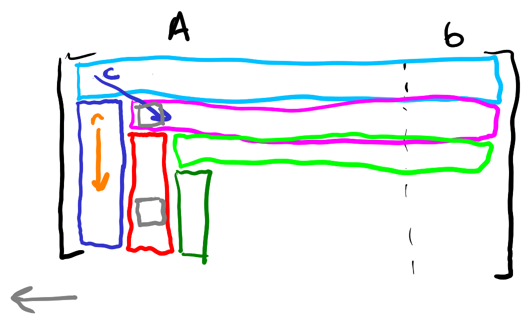
input : $n \times n$ matrix $[A]$, right-hand side $\{b\}$
output: solution $\{x\}$ of $[A]\{x\} = \{b\}$

Elimination

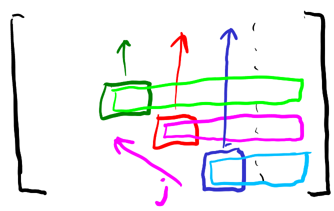
```
// forward elimination
for c ← 1 to n − 1 do
    for r ← c + 1 to n do
        f ← A[r, c] / A[c, c]
        A[r, c:n] ← A[r, c:n] − f · A[c, c:n]
        b[r] ← b[r] − f · b[c]
    end
end
```

Backsub

```
// back substitution
for r ← n, n − 1, . . . , 1 do
    s ← b[r]
    for j ← r + 1 to n do
        s ← s − A[r, j] · x[j]
    end
    x[r] ← s / A[r, r]
end
```



Back substitution

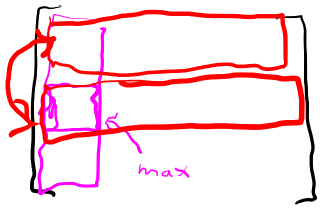


$$A = \begin{bmatrix} \boxed{0} & 2 & 3 \\ \boxed{4} & 6 & 7 \\ 2 & -3 & 6 \end{bmatrix} \qquad b = \begin{bmatrix} 8 \\ -3 \\ 5 \end{bmatrix}$$

$f \leftarrow \underline{Aug[1,2]} \,/\, \underline{Aug[1,1]}$ Divide by Zero

Solution: Partial Pivoting

input : $n \times n$ matrix $[A]$, right-hand side $\{b\}$
output: solution $\{x\}$ of $[A]\{x\} = \{b\}$



`// forward elimination`
for $c \leftarrow 1$ **to** $n-1$ **do**
  `// find the largest entry in column c`
  $p \leftarrow \text{argmax}(\text{abs}(A[c{:}n,\ c])) + c - 1$
  `// swap that row into the pivot position`
  $A[c,\ :] \leftrightarrow A[p,\ :]$
  $b[c] \leftrightarrow b[p]$
  for $r \leftarrow c+1$ **to** n **do**
    $f \leftarrow A[r,c] \,/\, A[c,c]$
    $A[r, c{:}n] \leftarrow A[r, c{:}n] - f \cdot A[c, c{:}n]$
    $b[r] \leftarrow b[r] - f \cdot b[c]$
  end
end

`// back substitution`
for $r \leftarrow n, n-1, \ldots, 1$ **do**
  $s \leftarrow b[r]$
  for $j \leftarrow r+1$ **to** n **do**
    $s \leftarrow s - A[r,j] \cdot x[j]$
  end
  $x[r] \leftarrow s \,/\, A[r,r]$
end