ASEN 3502 Computational Methods
Practice Exam 2 (2026)

Instructions: This practice exam is meant to show the style and scope of Exam 2; it is not an exhaustive list of what may be asked. The equation sheet distributed with the exam will be available. On the real exam, you will write your answers on a provided answer sheet. For this exam, use your own paper.

Question 1. (20 points) Consider $\dot{x} = -100x$ with $x(0) = 1$.

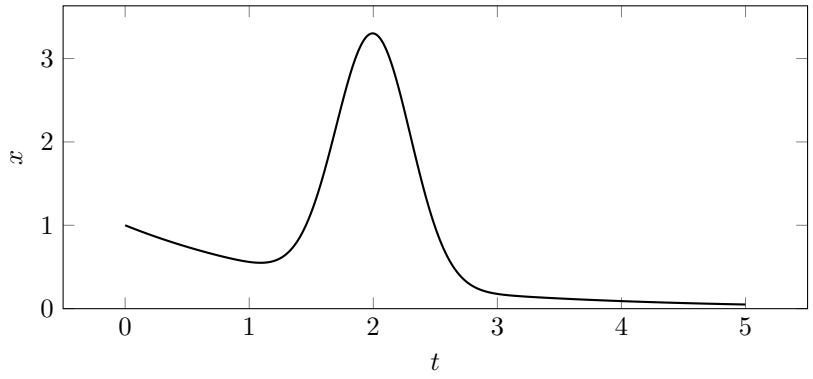
(a) What is the analytical solution, $x(t)$, for this initial value problem?

(b) Compute the result of one step of explicit Euler and one step of implicit Euler with $h = 0.05$.

(c) Compute the error, E_t, for each result.

(d) At this step size, will explicit or implicit Euler have smaller error after many steps? Justify your answer.

Question 2. (20 points) The solution of a certain ODE on $t \in [0, 5]$ is plotted below. You need to compute it numerically to high accuracy.



(a) Would it be better to use your fixed-step `rk4` from Lab 3 or `ode45`? Why? Which inputs to the function you chose control its accuracy?

(b) Matlab's `ode23` determines if the step size is appropriate by comparing two quantities calculated with Runge-Kutta integration. How do these two quantities differ? (you can describe with a few words and do not need to give an equation)

Question 3. (10 points) A projectile with position (z, y) experiences gravity and quadratic drag:

$$\ddot{z} = -c\, v\, \dot{z}, \qquad \ddot{y} = -g - c\, v\, \dot{y}, \qquad v = \sqrt{\dot{z}^2 + \dot{y}^2}.$$

Define a state vector $\vec{x}$ and write the equations as a first-order vector ODE $\dot{\vec{x}} = f(t, \vec{x})$. Then write f as a MATLAB anonymous function that could be passed to `ode45`.

Question 4. (10 points) Consider the boundary value problem

$$\ddot{x} = 0.05\,(x - 20), \qquad x(0) = 40, \qquad x(10) = 200.$$

You will solve this with the shooting method. Describe in words and symbols the residual function that can be given to a Lab 2 root finding function. In particular, what does this function take as input, what does it compute, what should it return, and when should it return zero.

Question 5. (20 points) Let $f(\vec{x}) = x_1^4 + (x_1 - x_2)^2 + x_2^2 - 5x_1$.

(a) Compute $\nabla f(\vec{x}_0)$ at $\vec{x}_0 = [0,\ 0]^T$.

(b) Take one gradient descent step with $\alpha = 1$. What is $\vec{x}_1$?

(c) Is this a good step size? Justify your answer.

Question 6. (20 points) $f(x) = x^4 - 3x^2 + x$ has a local minimum between $x = 1$ and $x = 2$. Write MATLAB code that finds it to within 10^{-6} using one of your Lab 2 root-finding functions (interface on the equation sheet).

Solutions

Question 1

(a) $\dot{x} = ax$ with $a = -100$ has solution $x(t) = Ce^{at}$; $x(0) = 1$ gives $C = 1$, so $x(t) = e^{-100t}$.

(b) Explicit: $x_1 = x_0 + (-100x_0)h = 1 - 5 = -4$. Implicit: $x_1 = x_0 + (-100x_1)h \Rightarrow x_1 = 1/(1+5) = 0.1667$.

(c) Exact: $x(0.05) = e^{-5} = 0.0067$. Errors: explicit $E_t = |-4 - 0.0067| = 4.007$, implicit $E_t = 0.160$.

(d) Implicit. Explicit Euler multiplies x by $1 - 100h = -4$ each step, so it is unstable at this h (the error grows exponentially; it would need $h < 2/100 = 0.02$). Implicit Euler multiplies by $1/(1 + 100h) = 1/6$ and is unconditionally stable for this type of ODE (as noted on the equation sheet), so its result at least decays like the true solution.

Question 2

(a) `ode45`. The solution changes rapidly only near $t = 2$ and slowly elsewhere. A fixed-step method needs a step small enough to resolve the spike *everywhere*, wasting most of its function evaluations on the smooth parts. `ode45` estimates the error at each step and shrinks h only near the spike, then grows it again. Its accuracy is controlled by `RelTol` and `AbsTol` set with `odeset` (for `rk4`, only h).

(b) One quantity is calculated with a second-order Runge-Kutta step, the other with a third-order step.

Question 3

$$\vec{x} = \begin{bmatrix} z \\ y \\ \dot{z} \\ \dot{y} \end{bmatrix}, \qquad f(t, \vec{x}) = \begin{bmatrix} \dot{z} \\ \dot{y} \\ -c\,v\,\dot{z} \\ -g - c\,v\,\dot{y} \end{bmatrix} \quad \text{or} \quad f(t, \vec{x}) = \begin{bmatrix} x_3 \\ x_4 \\ -c\sqrt{x_3^2 + x_4^2}\,x_3 \\ -g - c\sqrt{x_3^2 + x_4^2}\,x_4 \end{bmatrix}$$

```
f = @(t, x) [x(3); x(4); -c*norm(x(3:4))*x(3); -g - c*norm(x(3:4))*x(4)];
```

Question 4

Input: a guess v for the unknown initial rate $\dot{x}(0)$. The function then integrates $\dot{\vec{x}} = [\dot{x}, \ddot{x}]^T = [\dot{x},\ 0.05(x - 20)]^T$ from $\vec{x}(0) = [40,\ v]^T$ over $t \in [0, 10]$ with `rk4` or `ode45`. It returns the difference between the resulting estimate of $x(10)$ from the integration and 200. When $x(10)$ from the numerical integration equals the specified boundary value of 200, the residual function should return zero.

Question 5

(a) $\nabla f = \begin{bmatrix} 4x_1^3 + 2(x_1 - x_2) - 5 \\ -2(x_1 - x_2) + 2x_2 \end{bmatrix}$, so $\nabla f(\vec{x}_0) = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$.

(b) $\vec{x}_1 = \vec{x}_0 - \alpha \nabla f(\vec{x}_0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

(c) No. $f(\vec{x}_1) = 625 + 25 - 25 = 625$, while $f(\vec{x}_0) = 0$, so this step size does not reduce the objective.

Question 6

The minimum is a root of $f'(x) = 4x^3 - 6x + 1$, and $f'(1) = -1 < 0 < f'(2) = 21$, so $[1, 2]$ brackets it:

```
fp = @(x) 4*x^3 - 6*x + 1;
fpp = @(x) 12*x^2 - 6;
[x, info] = bisection(fp, fpp, [1, 2], 1e-6, 100);
```

`newton_raphson` or `secant` with the same arguments is also fine ($x^* \approx 1.137$; $f''(x^*) > 0$ confirms a minimum). Passing f itself to the root finder is wrong: it finds where $f = 0$, not where $f' = 0$.