ASEN 3502 Computational Methods
Homework 7

See Gradescope for Due Date

In order to practice for the exam, please do these problems **by hand** (a calculator is fine for the arithmetic); show your work and submit one pdf.

Question 1 (Standard form)
Write the general multivariable constrained optimization problem in standard form (as in class and in Martins & Ning, Ch. 5), with the decision variables collected in a vector $\vec{x}$. Then label each part of what you wrote:

(a) objective function
(b) decision variables
(c) equality constraints
(d) inequality constraints

Also state what it means for a point $\vec{x}$ to be *feasible*, and what it means for an inequality constraint to be *active* at $\vec{x}$.

Question 2 (Fuel tank design and the KKT conditions)
An external fuel tank is to be designed for a wing pylon. At this stage we approximate it as a circular cylinder of radius r and length l (the detailed aerodynamic shape will be designed later). It is made of sheet metal with areal density $s = 10$ kg/m^2 and holds fuel of density $\rho = 800$ kg/m^3 (neglect the thickness of the walls when computing the volume). We want to choose r and l to minimize the total mass of the full tank (fuel plus skin, including the two end caps), subject to the requirements that the tank hold at least $V_{\min} = 1$ m^3 of fuel and that the radius not exceed $r_{\max} = 0.4$ m for ground clearance.

(a) Write the problem in standard form with $\vec{x} = [r,\ l]^\mathsf{T}$: give $f(\vec{x})$ and $\vec{g}(\vec{x})$ (there are no equality constraints).

(b) State the KKT conditions for a problem with only inequality constraints, using multipliers σ_j for the g_j.

(c) Two candidate designs are proposed:

$$\text{A: } r = 0.542 \text{ m},\ l = 1.084 \text{ m} \qquad \text{B: } r = 0.4 \text{ m},\ l = 1.989 \text{ m}.$$

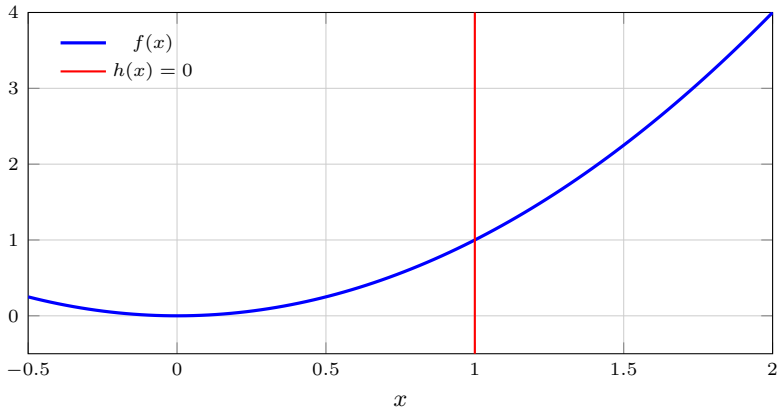
Evaluate $\vec{g}$ at each (small rounding error is fine). Which of these designs is feasible?

(d) For the feasible candidate, compute ∇f and the ∇g_j, then solve for the multipliers σ_1 and σ_2. Does the design satisfy the KKT conditions?

(e) ~~Interpret the multiplier on the radius constraint: if the ground clearance allowed a slightly larger radius, how much mass would be saved per meter of additional allowed radius?~~ (Removed: we did not have time to cover multiplier sensitivity in class. But, it is very cool! If you want to look it up, read Section 5.3.4 of Engineering Design Optimization by Martins and Ning.)

Question 3 (Augmented Lagrangian)
Consider the scalar equality-constrained problem (The same f and solution as the constrained optimization demo, but with the constraint written as an equality. The constraint trivially forces the solution to be $x^* = 1$, but the problem is useful for illustrating the augmented Lagrangian method.)

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) = x^2 \\ \text{subject to} \quad & h(x) = 1 - x = 0. \end{aligned}$$



(a) Find the exact solution x^* and the Lagrange multiplier λ^* by solving the KKT conditions $f'(x) + \lambda\, h'(x) = 0$, $h(x) = 0$.

(b) Write out the augmented Lagrangian

$$\hat{f}(x; \lambda, \mu) = f(x) + \lambda\, h(x) + \frac{\mu}{2}\, h(x)^2$$

for this problem and find $\partial \hat{f} / \partial x$.

(c) In the next two parts, we will complete one iteration of the augmented Lagrangian algorithm, computing x_0^* and x_1^*. Starting from $\lambda_0 = 0$ and $\mu_0 = 2$, find the minimizer x_0^* of $\hat{f}(x; \lambda_0, \mu_0)$ by setting $\partial \hat{f} / \partial x = 0$ (for this problem you can solve it exactly by hand). Evaluate $h(x_0^*)$.

(d) Update the multiplier and penalty parameter with

$$\lambda_1 = \lambda_0 + \mu_0\, h(x_0^*), \qquad \mu_1 = \rho\, \mu_0, \quad \rho = 2,$$

and find the next minimizer x_1^* of $\hat{f}(x; \lambda_1, \mu_1)$ and evaluate $h(x_1^*)$.

(e) Verify that the Lagrange multiplier estimate (λ_1) and minimum estimate (x_1^*) get closer to the true optimum than λ_0 and x_0^*.