ASEN 3502 Computational Methods
Homework 6

See Gradescope for Due Date

This homework gives practice with optimality conditions and the first steps of gradient-based optimization. In order to practice for the exam, please do these problems **by hand** (a calculator is fine for the arithmetic); show your work and submit one pdf.

Question 1 (Optimality conditions)

(a) For a C^2 continuous function $f(\vec{x})$, state the first-order necessary condition, the second-order necessary condition, and a second-order sufficient condition for $\vec{x}^*$ to be a local minimum.

(b) Give a scalar function $f(x)$ and a point x^* that satisfy both necessary conditions but where x^* is *not* a local minimum.

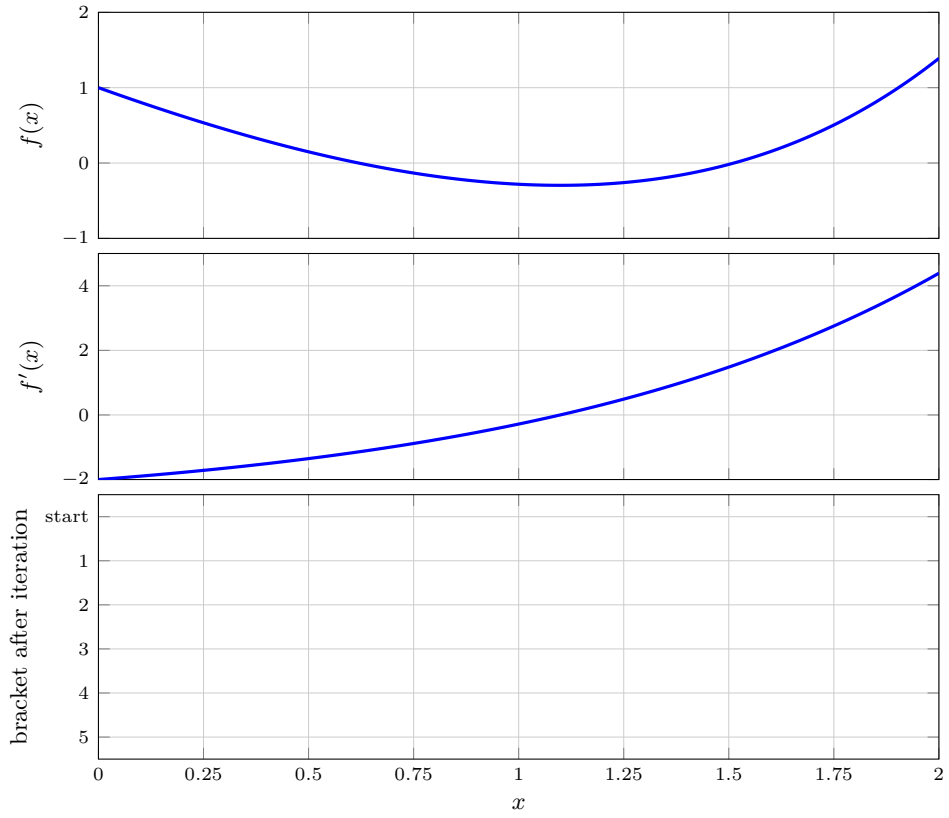
Question 2 (Bisection for a minimum)
Let $f(x) = e^x - 3x$. Graphs of f and f' on $[0, 2]$ are shown below.

(a) Find the minimizer x^* analytically and show that it is a minimum.

(b) Explain how bisection could be used to find x^*: what function would you give it, and why does $[0, 2]$ work as the initial bracket?

(c) Now run bisection *graphically*, reading signs off the graph of f' rather than evaluating it. Mark each function evaluation with a vertical line through the graph of f', and in the rows below draw the bracket after each iteration as a horizontal bar. Stop when the bracket is narrower than 0.1.

(d) How far is the midpoint of your final bracket from x^*?



Question 3 (Gradient descent and Newton's method)
Consider

$$f(\vec{x}) = x_1^4 + (x_1 - x_2)^2 + x_2^2 - 5x_1.$$

(a) Find $\nabla f(\vec{x})$ and the Hessian $H(\vec{x})$.

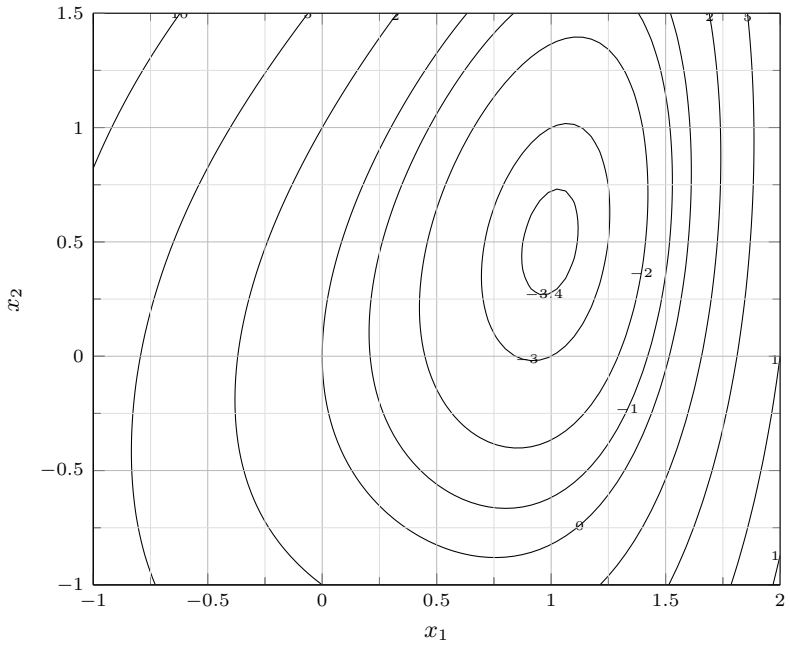
(b) Find the unique stationary point $\vec{x}^*$ and explain why it is the only one.

(c) Verify that $H(\vec{x}^*)$ is positive definite, so that $\vec{x}^*$ is a local minimum. *Hint:* a symmetric 2×2 matrix is positive definite if and only if its top-left entry and its determinant are both positive.

(d) Starting from $\vec{x}_0 = (0, 0)$, take one step of gradient descent, $\vec{p}_0 = -\nabla f(\vec{x}_0)$, choosing the step size with Armijo backtracking ($\alpha_{\text{init}} = 1$, $\mu_1 = 10^{-4}$, $\rho = 0.5$). Show each trial α and the two sides of the Armijo condition.

(e) Repeat from the same $\vec{x}_0$ with the Newton direction $\vec{p}_0 = -H(\vec{x}_0) \backslash \nabla f(\vec{x}_0)$ and the same line search.

(f) Contours of f are shown below. On your own sketch (a box with axes is enough; do not copy the contours), mark $\vec{x}_0$, $\vec{x}^*$, and the points reached in (d) and (e). Which step got closer to the minimum, and why does the full Newton step ($\alpha = 1$) fail here?



Question 4 (A constrained scalar problem)
The function below is from Martins & Ning, Ch. 4, Exercise 8; here we add a constraint. Consider

$$\underset{x}{\text{minimize}} \quad f(x) = -\frac{x}{x^2 + 2} \qquad \text{subject to} \quad x \leq 1.$$

(a) Ignoring the constraint, find the stationary points of f and classify each as a local minimum or maximum.

(b) Sketch $f(x)$ for $-4 \leq x \leq 4$ and shade the feasible region.

(c) Where is the constrained minimum x^*? What is $f(x^*)$, and is $f'(x^*) = 0$ there? Explain.