ASEN 3502 Computational Methods
Homework 5

See Gradescope for Due Date

This homework gives practice with implicit integration and the finite-difference method for boundary-value problems. In order to practice for the exam, please do these problems **by hand** (a calculator is fine for the arithmetic); show your work and submit one pdf.

Question 1 (Implicit Euler on a nonlinear equation)
Consider

$$\dot{y} = f(t, y) = -500\left(y^2 - \sin^2 t\right), \qquad y(0) = 1.$$

The large coefficient makes y change very rapidly at first, after which the solution varies much more slowly. This combination of fast and slow behavior makes the problem *stiff*. Use $h = 0.1$ throughout.

(a) Write the *implicit* Euler update for a general $\dot{y} = f(t, y)$ and explain why it is called "implicit."

(b) Take two steps of *explicit* Euler to obtain estimates of $y(0.1)$ and $y(0.2)$.

(c) Substitute this problem's f into the implicit Euler update and rearrange it into a quadratic equation in y_{i+1}.

(d) Take two steps of implicit Euler to obtain $y(0.1)$ and $y(0.2)$. Explain which root of the quadratic you chose and why (Hint: if $y = 0$, is $\dot{y}$ less than or greater than 0, and, based on that, will the solution cross $y = 0$?).

(e) An accurate reference solution gives $y(0.1) = 0.0880$ and $y(0.2) = 0.1935$. Compare both methods to these values. Which method yields a better estimate after 2 steps? Would it be better to describe the difference as an issue of accuracy or stability? Why?

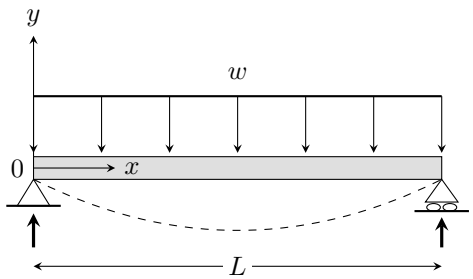
(f) If f were not quadratic in y (for example $-500(y^3 - \sin^3 t)$), describe in words how you would calculate the implicit Euler update with the aid of a computer and one or more algorithms that we have previously discussed in class.

Question 2 (Finite differences for a beam)
The deflection $y(x)$ of a simply supported, uniformly loaded beam (Chapra 24.16) satisfies

$$EI\,\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}, \qquad y(0) = y(L) = 0,$$

with $E = 200$ GPa, $I = 30{,}000$ cm^4, $w = 15$ kN/m, and $L = 3$ m. The analytical solution is

$$y(x) = \frac{wLx^3}{12EI} - \frac{wx^4}{24EI} - \frac{wL^3x}{24EI}.$$



Work in SI units (N, m) and report deflections in mm.

(a) Divide the beam into segments of length Δx with nodes $x_i = i\,\Delta x$. Replace d^2y/dx^2 with a centered finite-difference approximation and write the resulting linear equation relating y_{i-1}, y_i, and y_{i+1} at an interior node.

(b) Using $\Delta x = 1$ m (two interior nodes), write the system of equations in matrix form $[A]\{y\} = \{b\}$ with numerical entries, solve it by hand (use Gaussian elimination, Cramer's rule, or deduce a scalar equation based on symmetry), and compute the percent error relative to the analytical solution at each node.

(c) Using $\Delta x = 0.6$ m (four interior nodes), write the system in matrix form with numerical entries. Solve it (by hand, noting that symmetry about the midspan lets you reduce it to two unknowns, or with a computer) and compute the percent error at each node.

(d) As noted in Figure 21.5 from the book, the centered difference approximation has truncation error $O(\Delta x^2)$. Is the change in error between parts (b) and (c) consistent with this? Show your reasoning.